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THE W ALGEBRA STRUCTURE OF $N = 2$ CP_n COSET MODELS *

KATSUSHI ITO[†]*Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

ABSTRACT

We discuss the $N = 2$ super W algebras from the hamiltonian reduction of affine Lie superalgebras $A(n|n-1)^{(1)}$ and $A(n|n)^{(1)}$. From the quantum hamiltonian reduction of $A(n|n-1)^{(1)}$ we get the free field realization of $N = 2$ CP_n super coset models. In the case of the affine Lie superalgebras $A(n|n)^{(1)}$, the corresponding conformal field theories do not have $N = 2$ superconformal symmetry. However we show that these models are twisted $N = 2$ CP_n models and may be regarded as topological conformal field theories.

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[†]Address after March 1, 1993: Institute of Physics, University of Tsukuba, Ibaraki 305, Japan

1. Introduction

The W -algebra symmetry¹ in two dimensions plays a fundamental role in various integrable systems such as generalized KdV hierarchies and Toda field theories. In conformal field theories the W algebras provide an important class of chiral algebras and have been studied extensively². A systematic construction of the W -algebra associated with a simple Lie algebra is given by the hamiltonian reduction of the affine Lie algebra^{3,2}. One of the recent developments in this direction is that superconformal symmetries and their W -extensions are also obtained by considering the hamiltonian reduction of a certain class of affine Lie superalgebras⁴. In particular, the W -extension^{5–11} of the $N = 2$ superconformal symmetry is an interesting subject in view of its relation to topological field theories and the compactifications of superstrings.

In previous papers^{5,6}, the author has studied the quantum hamiltonian reduction of the affine Lie superalgebras $A(n|n-1)^{(1)}$ and the Feigin-Fuchs representations of the $N = 2$ super W algebras, which characterize the $N = 2$ super CP_n coset models constructed by Kazama and Suzuki¹² (see also ref. 7). Inami and Kanno⁹ observed that the classical $N = 2$ super W algebra appears also in the $N = 2$ super KdV hierarchies associated with the affine Lie superalgebras $A(n|n)^{(1)}$.

The purpose of the present talk is to explain why these different Lie superalgebras correspond to the same $N = 2$ super W algebra. We shall discuss the classical and quantum hamiltonian reduction of the affine Lie superalgebras $A(n|n-1)^{(1)}$ and $A(n|n)^{(1)}$. We will show that the conformal field theories associated with the affine Lie superalgebras $A(n|n)^{(1)}$, are the twisted $N = 2$ CP_n models.

2. Lie superalgebras and their hamiltonian reductions

2.1. Notations

Denote by \mathbf{g} a basic classical Lie superalgebra of rank r ¹³. $\mathbf{g} = \mathbf{h} \oplus (\oplus_{\alpha \in \Delta} \mathbf{g}_\alpha)$ is a root space decomposition of \mathbf{g} . \mathbf{h} is the Cartan subalgebra. The root system Δ of \mathbf{g} is $\Delta^0 \cup \Delta^1$, where Δ^0 (Δ^1) is the set of even (odd) roots. $\Delta_+ = \Delta_+^0 \cup \Delta_+^1$ is the set of positive roots of \mathbf{g} , where Δ_+^0 (Δ_+^1) is the set of even (odd) positive roots. $\mathbf{g}_0 = \mathbf{h} \oplus (\oplus_{\alpha \in \Delta^0} \mathbf{g}_\alpha)$ is an even subalgebra of \mathbf{g} . $\mathbf{g}_1 = \oplus_{\alpha \in \Delta^1} \mathbf{g}_\alpha$ is an odd subspace. ρ_0 (ρ_1) is half the sum of positive even (odd) roots and ρ is defined as $\rho_0 - \rho_1$. h^\vee is the dual Coxeter number of \mathbf{g} . For an affine Lie superalgebra $\hat{\mathbf{g}}$ at level k associated with \mathbf{g} , we define a constant $\alpha_+ = \sqrt{k + h^\vee}$. In the following we shall discuss the Lie superalgebras $A(m|n)$ ($h^\vee = m - n$) in detail.

2.2. The Lie superalgebras $A(m|n)$

A Lie superalgebra $sl(n+1|m+1)$ may be represented by matrices

$$X = \left(\begin{array}{c|c} A & a \\ \hline b & B \end{array} \right), \quad (1)$$

satisfying $\text{str}X = \text{tr}A - \text{tr}B = 0$, where A and B are $(n+1) \times (n+1)$ and $(m+1) \times (m+1)$ matrices with grassmann even elements and a and b are $(n+1) \times (m+1)$ and $(m+1) \times (n+1)$ matrices with grassmann odd elements. The commutation relation for two elements $X_i = \left(\begin{array}{c|c} A_i & a_i \\ \hline b_i & B_i \end{array} \right)$ ($i = 1, 2$) is given by

$$[X_1, X_2] = \left(\begin{array}{c|c} [A_1, A_2] + a_1b_2 - a_2b_1 & a_1B_2 - a_2B_1 + A_1a_2 - A_2a_1 \\ \hline B_1b_2 - b_1B_2 + b_1A_2 - b_2A_1 & [B_1, B_2] + b_1a_2 - b_2a_1 \end{array} \right). \quad (2)$$

In the case of $n = m$, the identity matrix $\mathbf{1}_{2n+2}$ spans an ideal of $sl(n+1|n+1)$ and the Lie superalgebra $A(n|n)$ is defined as $sl(n+1|n+1)/\langle \mathbf{1}_{2n+2} \rangle$. Even subalgebras of $A(n|m)$ for $n \neq m$ ($n = m$) are $A_n \oplus A_m \oplus u(1)$ ($A_n \oplus A_n$). For $A(n|n)$ it is convenient to use a pseudo-representation¹⁴. Namely we take a representative of elements of $A(n|n)$ in (1) such that $\text{tr}A = \text{tr}B = 0$, and modify the commutation relation like

$$[[X_1, X_2]] = [X_1, X_2] - \frac{1}{n+1} \text{tr}(a_1b_2 - a_2b_1) \mathbf{1}_{2n+2}. \quad (3)$$

In contrast to the simple Lie algebras, there is a variety of choices of the simple root system of Lie superalgebras, which correspond to different Dynkin diagrams. In the case of $m = n - 1$ and n , we may take the simple roots as purely odd roots. For $A(n|n-1)$ they are given by

$$\alpha_{2i-1} = e_i - \delta_i, \quad \alpha_{2i} = \delta_i - e_{i+1}, \quad i = 1, \dots, n, \quad (4)$$

where e_i ($i = 1, 2, \dots, n+1$) and δ_i ($i = 1, \dots, m+1$) are orthonormal bases with positive and negative metric. Similarly the simple roots of $A(n|n)$ are given by $\alpha_1, \dots, \alpha_{2n}, \alpha_{2n+1} = e_{n+1} - \delta_{n+1}$. The even positive roots of $A(n|m)$ ($m = n, n-1$) are $e_i - e_j$ ($1 \leq i < j \leq n+1$) and $\delta_i - \delta_j$ ($1 \leq i < j \leq m+1$). The odd positive roots are $e_i - \delta_j$ ($1 \leq i \leq j \leq m+1$) and $\delta_i - e_j$ ($1 \leq i < j \leq n+1$). In the matrix representation of the type (1), we have the fundamental representation of \mathfrak{g} :

$$\begin{aligned} E_{e_i - e_j} &= E_{i,j+1}, & E_{\delta_i - \delta_j} &= E_{n+1+i, n+2+j}, \\ E_{e_i - \delta_j} &= E_{i, n+1+j}, & E_{\delta_i - e_j} &= E_{n+1+i, j}, \end{aligned} \quad (5)$$

for positive roots. We define $E_{-\alpha} = {}^t E_{\alpha}$ for negative roots $-\alpha$ ($\alpha \in \Delta_+$). The Cartan elements are defined as $\alpha \cdot H = [E_{\alpha}, E_{-\alpha}]$. Instead of using (5), we can take a different representation^{5,6}, in which the odd simple root structure is manifest:

$$E_{e_i - e_j} = E_{2i-1, 2j-1}, \quad E_{\delta_i - \delta_j} = E_{2i, 2j}, \quad E_{e_i - \delta_j} = E_{2i-1, 2j}, \quad E_{\delta_i - e_j} = E_{2i, 2j-1}. \quad (6)$$

Note that in these expressions, $A(n|n-1)$ and $A(n|n)$ have a structure similar to that of the simple Lie algebras A_{2n+1} and A_{2n} , respectively.

The superalgebra $A(n|m)$ has rank $n + m + 1$, but the rank of $A(n|n)$ reduces to $2n$ due to the existence of the ideal. Moreover the root vectors are not linearly independent. In fact, from the relation $\sum_{i=1}^n [[E_{\alpha_{2i+1}}, E_{-\alpha_{2i+1}}]] = 0$, we have

$$\alpha_1 + \alpha_3 + \dots + \alpha_{2n+1} = 0. \quad (7)$$

For both Lie superalgebras $A(n|n-1)$ and $A(n|n)$ half the sum of positive roots ρ may be shown to be zero.

2.3. The classical hamiltonian reduction

Let us consider the hamiltonian reduction of the affine Lie superalgebras $\hat{\mathfrak{g}} = A(n|n-1)^{(1)}$ and $A(n|n)^{(1)}$. On the dual space $\hat{\mathfrak{g}}^*$ of $\hat{\mathfrak{g}}$ one may introduce the hamiltonian structure which is generated by the coadjoint action or the gauge transformation:

$$\delta_\Lambda J(z) = [\Lambda(z), J(z)] + \partial\Lambda(z). \quad (8)$$

Let $J_\alpha(z)$ ($\alpha \in \Delta$) and $H^i(z)$ ($i = 1, \dots, r$) be the currents in the canonical basis. We consider the constraint space \mathcal{M} in $\hat{\mathfrak{g}}^*$:

$$J_{\alpha_i + \alpha_{i+1}}(z) = 1, \quad J_{\alpha_i + \dots + \alpha_j}(z) = 0, \quad \text{for } |i - j| > 1. \quad (9)$$

In the present case we treat a dynamical system with second class constraints¹⁵ since the fermionic currents J_{α_i} must satisfy $J_{\alpha_i}(z)J_{\alpha_{i+1}}(w) \sim 1/(z-w)$. By introducing auxiliary fermionic fields one can convert these second class constraints into first class constraints¹⁶. In this extended phase space, one may use the ordinary gauge fixing procedure. The first class constraints generate the gauge symmetries in the extended phase space. One finds that these gauge symmetries are generated by the subalgebra $\hat{\mathfrak{n}}_-$, the affine extension of the nilpotent subalgebra \mathfrak{n}_- generated by the negative roots. The hamiltonian structure on the reduced phase space $\mathcal{M}/\hat{\mathfrak{n}}_-$ is introduced by projecting the original gauge symmetries onto the reduced phase space¹⁷. This method is also known as Polyakov's soldering procedure¹⁸. One may take the Drinfeld-Sokolov type gauge for $A(n|n-1)^{(1)}$:

$$\begin{aligned} J_{DS}(z) = & \sum_{i=1}^{n+1} U_{n+2-i}(z)E_{n+1,i} + \sum_{i=1}^n V_{n+1-i}(z)E_{2n+1,n+1+i} \\ & + \sum_{i=1}^n (G_{n+1-i}(z)E_{n+1,n+1+i} + \bar{G}_{n+1-i}(z)E_{2n+2,i}) + \Lambda^{A(n|n-1)} \end{aligned} \quad (10)$$

where

$$\Lambda^{A(n|n-1)} = \sum_{i=1}^n E_{i,i+1} + \sum_{i=1}^{n-1} E_{n+1+i,n+2+i} \quad (11)$$

and $V_1 = U_1$. For $A(n|n)^{(1)}$, we can choose a similar type of gauge fixing. It is obtained simply by replacing $\Lambda^{A(n|n-1)}$ by

$$\Lambda^{A(n|n)} = \sum_{i=1}^n E_{i,i+1} + \sum_{i=1}^n E_{n+1+i,n+2+i}. \quad (12)$$

3. The classical $N = 2$ Super W_3 Algebra

In this section we shall give a non-trivial example of the $N = 2$ super W -algebra from the classical hamiltonian reduction of the affine Lie superalgebra $A(2|1)^{(1)}$. Let

us take the Drinfeld-Sokolov gauge

$$J_{DS}(z) = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ U_3(z) & U_2(z) & U_1(z) & G_2(z) & G_1(z) \\ \hline 0 & 0 & 0 & 0 & 1 \\ \bar{G}_2(z) & \bar{G}_1(z) & 0 & V_2(z) & U_1(z) \end{array} \right). \quad (13)$$

The gauge transformation (8) has the gauge parameter

$$\Lambda(z) = \left(\begin{array}{ccc|cc} x_{11}(z) & x_{12}(z) & x_{13}(z) & \xi_{11}(z) & \xi_{12}(z) \\ x_{21}(z) & x_{22}(z) & x_{23}(z) & \xi_{21}(z) & \xi_{22}(z) \\ x_{31}(z) & x_{32}(z) & x_{33}(z) & \xi_{31}(z) & \xi_{32}(z) \\ \hline \eta_{11}(z) & \eta_{12}(z) & \eta_{13}(z) & y_{11}(z) & y_{12}(z) \\ \eta_{21}(z) & \eta_{22}(z) & \eta_{23}(z) & y_{21}(z) & y_{22}(z) \end{array} \right), \quad (14)$$

where the diagonal elements are parametrized as $x_{11} = \varepsilon_1 + 2\varepsilon$, $x_{22} = \varepsilon_2 - \varepsilon_1 + 2\varepsilon$, $x_{33} = -\varepsilon_2 + 2\varepsilon$ and $y_{11} = \varepsilon_3 + 3\varepsilon$ and $y_{22} = -\varepsilon_3 + 3\varepsilon$. By imposing the conditions that the gauge transformations preserve the Drinfeld-Sokolov gauge (13), one can reduce the number of independent gauge parameters and find that all gauge parameters are expressed in terms of $x_1 \equiv x_{13}$, x_{23} , y_{12} , ε , $\eta_1 \equiv \eta_{13}$, $\eta_2 \equiv \eta_{23}$, $\xi_1 \equiv \xi_{12}$ and $\xi_2 \equiv \xi_{22}$.

Note that we may regard the bosonic part of this system as the coupled one of $gl(3)$ and $gl(2)$ W algebras which share the same $u(1)$ current U_1 . But if one requires the $N = 2$ superconformal symmetry which is generated by ξ_2 and η_2 , one finds that the $N = 2$ supermultiplet of the $u(1)$ current U_1 is $(U_1, G_1, \bar{G}_1, U_2 - V_2)$. Hence $T \equiv U_2 - V_2$ becomes the energy-momentum tensor, and V_2 turns out to be a spin two field. Corresponding to this change of physical variables, we redefine the gauge parameters as $x = x_{23} + y_{12}$ and $y = y_{12}$ instead of x_{23} and y_{12} . One may add suitable differential polynomials of fields in order to get well defined primary fields. This can be done by replacing a gauge parameter ε by

$$\varepsilon + \frac{1}{6} \{U_1 x + 3\partial x - 3U_1 y - 3\partial y + 2U_2 x_1 - 2\partial(U_1 x_1) - 2\partial^2 x_1 + 3G_1 \eta_1 - 2\bar{G}_1 \xi_1\}. \quad (15)$$

After this change of variables, one gets the gauge transformations on the reduced phase space of the form:

$$\delta X_i = \mathcal{D}_{ij} Y_j, \quad (16)$$

where $X = {}^t (U_1, T, V_2, U_3, G_1, G_2, \bar{G}_1, \bar{G}_2)$, $Y = {}^t (\varepsilon, x, y, x_1, \xi_1, \xi_2, \eta_1, \eta_2)$ and \mathcal{D} is a 8×8 matrix valued differential operator which satisfies $\mathcal{D}_{ji} = \mathcal{D}_{ij}^*$, where for a differential operator $\sum_i a_i(z) \partial^i$, a formal adjoint $*$ is defined as $(\sum_i a_i(z) \partial^i)^* = \sum_i (-\partial)^i a_i(z)$.

Let us introduce the Poisson bracket structure by expressing δ_Λ as

$$\delta_\Lambda = \int \frac{dz}{2\pi i} \text{str} \left\{ \left(\begin{array}{ccc|cc} 2\varepsilon & 0 & x_1 & 0 & \xi_1 \\ 0 & 2\varepsilon & x - y & 0 & \xi_2 \\ 0 & 0 & 2\varepsilon & 0 & 0 \\ \hline 0 & 0 & \eta_1 & 3\varepsilon & y \\ 0 & 0 & \eta_2 & 0 & 3\varepsilon \end{array} \right) \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ U_3 & T + V_2 & U_1 & G_2 & G_1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \bar{G}_2 & \bar{G}_1 & 0 & V_2 & U_1 \end{array} \right) \right\}$$

$$= \int \frac{dz}{2\pi i} (U_3 x_1 + T x - y V_2 - \varepsilon U_1 + \xi_1 \bar{G}_2 + \xi_2 \bar{G}_1 - \eta_1 G_2 - \eta_2 G_1). \quad (17)$$

From Eqs. (16) and (17), we get the Poisson bracket structure on the reduced phase space in the form of operator product expansions. First we write down the $N = 2$ superconformal algebra:

$$\begin{aligned} T(z)T(w) &\sim \frac{T(z) + T(w)}{(z-w)^2}, & T(z)U_1(w) &\sim \frac{6}{(z-w)^3} + \frac{U_1(z)}{(z-w)^2}, \\ \bar{G}_1(z)T(w) &\sim \frac{\bar{G}_1(z) + \bar{G}_1(w)}{(z-w)^2}, & G_1(z)T(w) &\sim \frac{G_1(w)}{(z-w)^2}, \\ U_1(z)U_1(w) &\sim \frac{-6}{(z-w)^2}, \\ U_1(z)G_1(w) &\sim \frac{G_1(w)}{(z-w)}, & U_1(z)\bar{G}_1(w) &\sim \frac{-\bar{G}_1(w)}{(z-w)}, \\ G_1(z)\bar{G}_1(w) &\sim \frac{-6}{(z-w)^3} + \frac{U_1(w)}{(z-w)^2} + \frac{T(w)}{(z-w)}. \end{aligned} \quad (18)$$

Note that in this expression $T(z)$ is the twisted $N = 2$ energy-momentum tensor. Second the $N = 2$ supermultiplet structures for $(V_2, G_2, \bar{G}_2, U_3)$ are given by

$$\begin{aligned} T(z)V_2(w) &\sim \frac{6}{(z-w)^4} + \frac{-U_1(w)}{(z-w)^3} + \frac{V_2(w) + V_2(z)}{(z-w)^2}, \\ T(z)U_3(w) &\sim \frac{U_3(z) + 2U_3(w)}{(z-w)^2}, \\ T(z)\bar{G}_2(w) &\sim \frac{2\bar{G}_2(w) + \bar{G}_2(z)}{(z-w)^2}, & T(z)G_2(w) &\sim \frac{-2G_1(w)}{(z-w)^3} + \frac{G_2(w) + G_2(z)}{(z-w)^2}, \\ U_1(z)V_2(w) &\sim \frac{-6}{(z-w)^3} + \frac{3U_1(w)}{(z-w)^2}, \\ U_1(z)U_3(w) &\sim \frac{-12}{(z-w)^4} + \frac{4U_1(w)}{(z-w)^3} + \frac{2(T + V_2)(w)}{(z-w)^2}, \\ U_1(z)G_2(w) &\sim \frac{3G_1(w)}{(z-w)^2} + \frac{G_2(w)}{(z-w)}, & U_1(z)\bar{G}_2(w) &\sim \frac{2\bar{G}_1(w)}{(z-w)^2} + \frac{-\bar{G}_2(w)}{(z-w)}, \\ \bar{G}_1(z)V_2(w) &\sim \frac{\bar{G}_1(w)}{(z-w)^2} + \frac{\bar{G}_2(w)}{(z-w)}, & G_1(z)V_2(w) &\sim \frac{-G_2(w)}{(z-w)}, \\ \bar{G}_1(z)U_3(w) &\sim \frac{2\bar{G}_2(w) + \bar{G}_2(z)}{(z-w)^2}, & G_1(z)U_3(w) &\sim \frac{4G_1(w)}{(z-w)^3} + \frac{2G_2(w)}{(z-w)^2}, \\ \bar{G}_2(z)G_1(w) &\sim \frac{-12}{(z-w)^4} + \frac{-4U_1(z)}{(z-w)^3} + \frac{2V_2(z)}{(z-w)^2} + \frac{-U_3(w)}{(z-w)}, \\ \bar{G}_1(z)G_2(w) &\sim \frac{6}{(z-w)^4} + \frac{-2U_1(w)}{(z-w)^3} + \frac{V_2(w) + V_2(z) + T(w)}{(z-w)^2} + \frac{-U_3(w)}{(z-w)}. \end{aligned} \quad (19)$$

Finally the remaining nontrivial operator product expansions take the forms:

$$\begin{aligned}
V_2(z)V_2(w) &\sim \frac{-12}{(z-w)^4} + \frac{4[U_1(w) - U_1(z)]}{(z-w)^3} + \frac{V_2(z) + V_2(w)}{(z-w)^2} + \frac{2U_1(z)U_1(w)}{(z-w)}, \\
U_3(z)V_2(w) &\sim \frac{-24}{(z-w)^5} + \frac{6[U_1(w) - U_1(z)]}{(z-w)^4} + \frac{2[U_2(z) + U_1(z)U_1(w)]}{(z-w)^3} \\
&\quad + \frac{-U_1(w)U_2(z) + \bar{G}_1(w)G_1(z)}{(z-w)^2} + \frac{[\bar{G}_2G_1 - \bar{G}_1G_2](w)}{(z-w)}, \\
U_3(z)U_3(w) &\sim \frac{2[U_3(z) - U_3(w)]}{(z-w)^3} \\
&\quad + \frac{-U_3(w)U_1(z) - U_1(w)U_3(z) + \bar{G}_2(w)G_1(z) - G_1(w)\bar{G}_2(z)}{(z-w)^2}, \\
\bar{G}_2(z)V_2(w) &\sim \frac{2[\bar{G}_1(z) - \bar{G}_1(w)]}{(z-w)^3} \\
&\quad + \frac{-\bar{G}_2(w) - \bar{G}_1(w)U_1(z) - U_1(w)G_1(z)}{(z-w)^2} + \frac{[\bar{G}_1V_2 - \bar{G}_2U_1](w)}{(z-w)}, \\
G_2(z)V_2(w) &\sim \frac{4G_1(z)}{(z-w)^3} + \frac{-2U_1(w)G_1(z) - G_2(z)}{(z-w)^2} + \frac{[U_1G_2 - V_2G_1](w)}{(z-w)}, \\
\bar{G}_2(z)U_3(w) &\sim \frac{2[\bar{G}_2(z) - \bar{G}_2(w)]}{(z-w)^3} - \frac{U_1(w)\bar{G}_2(z) + \bar{G}_2(w)U_1(z)}{(z-w)^2} + \frac{[U_3\bar{G}_1 - T\bar{G}_2](w)}{(z-w)}, \\
G_2(z)U_3(w) &\sim \frac{6[G_1(z) - G_1(w)]}{(z-w)^4} + \frac{-2G_2(w) - 2G_1(w)U_1(z) - 2U_1(w)G_1(z)}{(z-w)^3} \\
&\quad + \frac{-G_2(w)U_1(z) - G_1(w)T(z) - U_2(w)G_1(z)}{(z-w)^2} + \frac{-[G_2T + G_1U_3](w)}{(z-w)}, \\
\bar{G}_2(z)G_2(w) &\sim \frac{-24}{(z-w)^5} + \frac{6[U_1(w) - U_1(z)]}{(z-w)^4} + \frac{2[U_1(w)U_1(z) + V_2(z) + T(w)]}{(z-w)^3} \\
&\quad + \frac{U_3(w) + T(w)U_1(z) - U_1(w)V_2(z) + G_1(w)\bar{G}_1(z)}{(z-w)^2} + \frac{[U_3U_1 - V_2T](w)}{(z-w)}, \\
G_2(z)G_2(w) &\sim \frac{[2G_1\partial G_1 + 2G_2G_1](w)}{(z-w)}, \quad \bar{G}_2(z)\bar{G}_2(w) \sim \frac{2\bar{G}_1\bar{G}_2(w)}{(z-w)}. \tag{20}
\end{aligned}$$

The present Poisson bracket structure is the same as that obtained from the super Gel'fand-Dickii algebra^{9,10} but different from the results in ref. 8 due to the different choice of the gauges.

4. The quantum hamiltonian reduction and $N = 2$ CP_n coset models

So far we have discussed the classical hamiltonian reduction. In the quantum case, we use the BRST gauge fixing procedure by introducing ghost systems for the constraints^{3,2}. In order to impose the constraints at the quantum level, we must

improve the energy-momentum tensor T_{WZNW} of the Wess-Zumino-Novikov-Witten model corresponding to the affine Lie superalgebra $\hat{\mathfrak{g}}$ by the Cartan currents $H^i(z)$,

$$T_{improved}(z) = T_{WZNW}(z) + \mu \cdot \partial H(z). \quad (21)$$

After this improvement, the conformal dimensions of the currents $J_\alpha(z)$ become $1 - \mu \cdot \alpha$.

First we discuss the $A(n|n-1)^{(1)}$ case. The improvement vector $\mu = \mu_{A(n|n-1)}$ is defined by the conditions:

$$\alpha_i \cdot \mu_{A(n|n-1)} = \frac{1}{2}, \quad i = 1, \dots, 2n. \quad (22)$$

After this improvement, the conformal dimensions of the currents for the odd simple roots $J_{\alpha_i}(z)$ becomes $\frac{1}{2}$ and zero for the the currents for the simple roots $\alpha_i + \alpha_{i+1}$ of the even subalgebras. Hence we may introduce the “diagonal” gauge:

$$J_{diag}(z) = \partial\varphi(z) \cdot H + \sum_{i=1}^n (\alpha_{2i-1} \cdot \chi(z) E_{\alpha_{2i-1}} + \alpha_{2i} \cdot \chi(z) E_{\alpha_{2i}}) + \Lambda^{A(n|n-1)}, \quad (23)$$

where $\chi^i(z)$ ($i = 1, \dots, 2n$) are $2n$ real fermions, satisfying $\chi^i(z)\chi^j(w) \sim \delta_{ij}/(z-w)$. The improvement vector $\mu_{A(n|n-1)}$ is uniquely determined by (22) and is expressed as

$$\mu_{A(n|n-1)} = \frac{1}{2} \sum_{i=1}^n [(n+1-i)\alpha_{2i-1} + i\alpha_{2i}] = \frac{1}{2} \sum_{i=1}^{2n} \lambda_i, \quad (24)$$

where λ_i are fundamental weights of $A(n|n-1)$ satisfying $\lambda_i \cdot \alpha_j = \delta_{ij}$:

$$\lambda_{2i} = \alpha_1 + \alpha_3 + \dots + \alpha_{2i-1}, \quad \lambda_{2i-1} = \alpha_{2i} + \alpha_{2i+2} + \dots + \alpha_{2n}, \quad (25)$$

for $i = 1, \dots, n$. By using the Wakimoto realization of affine Lie superalgebras, one finds that the total energy-momentum tensor $T_{total} = T_{improved} + T_{ghosts} + T_\chi$ is BRST-equivalent to that of the $N = 2$ CP_n coset model^{5,7}:

$$T_{total}(z) = T_{CP_n}(z) + \{Q_{BRST}, *\}, \quad (26)$$

where $T_\chi = -\frac{1}{2} \sum_{i=1}^{2n} \chi^i \partial \chi^i$ and

$$T_{CP_n}(z) = -\frac{1}{2}(\partial\varphi)^2 - i\alpha_+ \mu_{A(n|n-1)} \cdot \partial^2 \varphi + T_\chi. \quad (27)$$

The remaining $N = 2$ generators are expressed as

$$\begin{aligned} U_1(z) &= \sum_{i=1}^n \lambda_{2i} \cdot \chi \alpha_{2i} \cdot \chi + i\alpha_+ \nu \cdot \partial\varphi, \\ G_1(z) &= \sum_{j=1}^n (i\alpha_{2j} \cdot \partial\varphi \lambda_{2j} \cdot \chi - \alpha_+ \lambda_{2j} \cdot \partial\chi), \\ \bar{G}_1(z) &= \sum_{j=1}^n (i\alpha_{2j-1} \cdot \partial\varphi \lambda_{2j-1} \cdot \chi - \alpha_+ \lambda_{2j-1} \cdot \partial\chi), \end{aligned} \quad (28)$$

where $\nu = \sum_{i=1}^n (\lambda_{2i} - \lambda_{2i-1})$. The other W currents can be obtained by the quantum Miura transformation, which connects the diagonal gauge (23) with the Drinfeld-Sokolov gauge (10). The free field representation of the $N = 2$ CP_n models has been studied in ref. 6 in detail.

Now we proceed to the affine Lie superalgebra $A(n|n)^{(1)}$. In this case one cannot impose the spin $\frac{1}{2}$ constraints (22) for the simple roots α_i ($i = 1, \dots, 2n+1$) due to the relation (7). Instead we should require the conditions:

$$\alpha_{2i-1} \cdot \mu = 0, (i = 1, \dots, n+1), \quad \alpha_{2j} \cdot \mu = 1, (j = 1, \dots, n), \quad (29)$$

that are consistent with (7). This means that the conformal weights are 1 for $J_{\alpha_{2i-1}}$ but 0 for $J_{\alpha_{2i}}$ after the improvement. The vector μ is determined uniquely up to $\sum_{i=1}^{n+1} \alpha_{2i-1} (= 0)$:

$$\mu = \mu_{A(n|n)} = \sum_{i=1}^{n+1} (n+1-i) \alpha_{2i-1}. \quad (30)$$

The diagonal gauge for $A(n|n)^{(1)}$ is

$$J_{diag}(z) = \partial\varphi(z) \cdot H + \sum_{i=1}^{n+1} \eta_{2i-1}(z) E_{\alpha_{2i-1}} + \sum_{i=1}^n \xi_{2i}(z) E_{\alpha_{2i}} + \Lambda^{A(n|n)}, \quad (31)$$

where $\eta_{2i-1} = \eta_{2i} - \eta_{2i-2}$ ($\eta_{2n+2} = \eta_0 \equiv 0$) and (η_{2i}, ξ_{2i}) are fermionic ghosts with conformal weight $(1, 0)$. The energy-momentum tensor of the reduced theory becomes

$$\tilde{T}_{CP_n}(z) = -\frac{1}{2}(\partial\varphi)^2 - i\alpha_+ \mu_{A(n|n)} \cdot \partial^2\varphi - \sum_{i=1}^n \eta_{2i} \partial \xi_{2i}, \quad (32)$$

which is equal to $T_{CP_n} + \frac{1}{2}\partial U_1$. This is nothing but the energy-momentum tensor of the twisted¹⁹ $N = 2$ CP_n model. Hence the conformal field theory corresponding to the affine Lie superalgebra $A(n|n)^{(1)}$ should be regarded as a topological conformal field theory rather than an $N = 2$ superconformal field theory. This implies that the $A(n|n)$ Toda field theory has the *twisted* $N = 2$ super W -algebra symmetry instead of the $N = 2$ superconformal symmetry. Recently Evans and Hollowood²⁰ also pointed out that the $A(n|n)$ Toda field theory does not have $N = 2$ superconformal symmetry.

In this sense the $A(n|n)^{(1)}$ affine Toda field theory can be also regarded as a topological field theory rather than an $N = 2$ theory. It is a quite interesting problem to study this affine Toda field theory as a topological field theory since this model gives a different class of topological solvable models which are not classified by the $N = 2$ Landau-Ginzburg type models.

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